# Schwarz Lemma and non-Euclidean Geometry

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Hermann Schwarz

Theorem (Schwarz Lemma). ~~Schwartz Lemma.~~

Let $f\in A(\mathbb{D})$, $|f(z)|\le 1$ $\forall z\in\mathbb{D}$, $f(0)=0$.

Then $\forall z\in\mathbb{D}$ $|f(z)|\le|z|$. and $|f'(0)|\le 1$.

If for some $z\in\mathbb{D}\setminus\{0\}$ $|f(z)|=|z|$ or $|f'(0)|=1$

then $\exists\theta: f(z)=e^{i\theta}z$. ($f$ is a rotation by $\theta$).

Proof. Let $\varphi(z):=\begin{cases}\frac{f(z)}{z}, & z\ne 0\\ f'(0), & z=0.\end{cases}$

Then $\varphi\in A(\mathbb{D}\setminus\{0\})$, $\lim_{z\to 0}\varphi(z)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}=f'(0)$, so $\varphi\in A(\mathbb{D})$.

Take $r<1$. Then, by maximum principle $\forall z: |z|<r$:

$$|\varphi(z)|\le\max_{|z|=r}|\varphi(z)|=\max_{|z|=r}\frac{|f(z)|}{r}\le\frac{1}{r}.$$

So $\forall z: |z|<1$ we have $|\varphi(z)|\le 1 \Rightarrow \left|\frac{f(z)}{z}\right|\le 1$

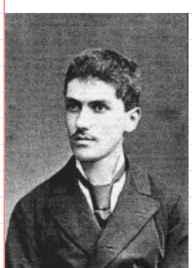
$|f'(0)|\le 1$.

If for some $z$, $|\varphi(z)|=1$ (which is $|f(z)|=|z|$ if $z\ne 0$, $|f'(0)|=1$, $z=0$)

then $|\varphi|$ reaches maximum at $z$, so

$\varphi(z)=\text{const}$. $|\text{const}|=1\Rightarrow \text{const}=e^{i\theta}$

$\frac{f(z)}{z}=e^{i\theta}$ ■



Georg Pick

An invariant form of Schwarz Lemma.

Theorem. (Schwarz-Pick).

Let $f\in A(\mathbb{D})$, $f:\mathbb{D}\to\mathbb{D}$ (i.e. $\forall z\in\mathbb{D}$: $|f(z)|<1$).

Then $\forall z_1, z_2\in\mathbb{D}$ $\forall z\in\mathbb{D}$

$$\frac{|f(z_1)-f(z_2)|}{|1-\overline{f(z_1)}f(z_2)|} \le \frac{|z_1-z_2|}{|1-\bar z_1 z_2|} \quad \text{and} \quad \frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}$$

If the equality is reached <u>for some</u> $z_1, z_2 \in \mathbb{D}$, <u>or</u> <u>for some</u> $z \in \mathbb{D}$, then $f$ is a Möbius transformation $\mathbb{D} \to \mathbb{D}$.

<u>Proof</u>. For $w \in \mathbb{D}$, denote $S_w(z) = \frac{z-w}{1-\bar w z}$. $S_w(w)=0$.

Consider the map $g(z) := S_{f(z_1)} \circ f \circ S_{z_1}^{-1}$. Then

$g(0) = S_{f(z_1)} \circ f \circ S_{z_1}^{-1}(0) = S_{f(z_1)} f(z_1) = 0$, and $g: \mathbb{D} \to \mathbb{D}$

<u>So, by Schwarz lemma:</u> (since each map does it).

$|S_{f(z_1)} \circ f \circ S_{z_1}^{-1}(z)| \le |z| \quad \forall z \in \mathbb{D}$. Take $z = S_{z_1}(z_2) = \frac{z_2 - z_1}{(1-\bar z_1 z_2)}$.

Then $S_{z_1}^{-1}(z) = z_2$, $\quad S_{f(z_1)} \circ f \circ S_{z_1}^{-1}(z) = S_{f(z_1)}(f(z_2)) = \frac{f(z_1)-f(z_2)}{1-\overline{f(z_1)}f(z_2)}$.

So it implies the inequality.

Rewrite it: $\displaystyle \frac{|f(z_1)-f(z_2)|}{|z_1-z_2|} \le \frac{|1-\overline{f(z_1)}f(z_2)|}{|1-\bar z_1 z_2|}$

Let $z_2 \to z_1$ to get the second inequality.

Finally, equality is reached in any of the inequalities $\iff$

$|g(z)| = |z|$ for some $z$ or $|g'(0)| = 1 \iff g(z) = e^{i\varphi} z \iff$

$f = S_{f(z_1)}^{-1} \circ g \circ S_{z_1}$ – Möbius $\blacksquare$

<u>Def</u>. $\displaystyle \rho(z_1, z_2) := \left| \frac{z_1 - z_2}{1-\bar z_1 z_2} \right| = \sqrt{\langle z_1, \frac{1}{\bar z_1}, z_2, \frac{1}{\bar z_2} \rangle}$

quasihyperbolic metric.

Möbius maps fixing circle preserve:

1) Cross ratio.
2) Points symmetric with respect to the unit circle.

So they preserve $\rho$: if $f = e^{i\theta}\frac{z-a}{1-\bar a z}$ then $\rho(f(z_1), f(z_2)) = \rho(z_1, z_2)$.

Why is $\rho$ metric?

$\left.\begin{array}{l} \rho(z_1,z_2) = 0 \iff z_1 = z_2 \\ \rho(z_1,z_2) = \rho(z_2,z_1) \end{array}\right\}$ Obvious

$\rho(z_1, z_2) + \rho(z_2, z_3) \ge \rho(z_1, z_3)$.

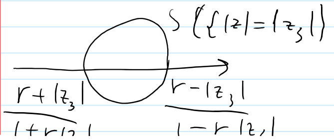
Möbius-invariant, so map $z_2$ to $0$, $z_1$ to $r > 0$

Then $\rho(r, 0) = r$, $\quad \rho(z_3, 0) = |z_3|$, $\quad \rho(r, z_3) = \frac{|r - z_3|}{|1 - r z_3|}$

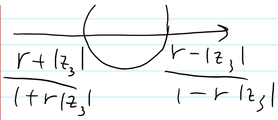
For fixed $r$, the image of the circle $\{|z| = |z_3|\}$ under $S(z) := \frac{r - z}{1 - rz}$

a circle symmetric wrt $\mathbb{R}$, $|S(|z_3|)| = \left|\frac{r - |z_3|}{1 - r|z_3|}\right| \le r + |z_3|$

$S(-|z_3|) = \frac{r + |z_3|}{1 + r|z_3|} \le r + |z_3|$.




So $\frac{|r - z_3|}{\phantom{x}} \le r + |z_3|$




$$\phi(-|z_3|) = \frac{r+|z_3|}{1+r|z_3|} \le r+|z_3|.$$

So $$\frac{|r-z_3|}{|1-rz_3|} \le r+|z_3|.$$

**Theorem.** Let $f \in A(\mathbb{D})$, $f: \mathbb{D} \to \mathbb{D}$ – bijection. Then $f$ is a Möbius map.

**Proof.** $\forall z_1, z_2 \in \mathbb{D}$.

$$\rho(f(z_1), f(z_2)) \le \rho(z_1, z_2)$$

But $f^{-1}: \mathbb{D} \to \mathbb{D}$, analytic.

So $\rho(z_1,z_2) = \rho(f^{-1}(f(z_1)), f^{-1}(f(z_2))) \le \rho(f(z_1), f(z_2))$.

So $\rho(z_1, z_2) = \rho(f(z_1), f(z_2)) \Rightarrow f$ is Möbius. ∎

**Corollary.** $\rho$ is invariant under all conformal bijections of $\mathbb{D}$ to itself.

## Hyperbolic metric.

How to measure the length of curve?

$$\ell(\gamma) = \int_a^b |z'(t)|\,dt$$ How to measure it.

Know: under Möbius, $\frac{|f'(z)|}{1-|f(z)|^2} = \frac{1}{1-|z|^2}$. By Schwarz-Pick, for any $f \in A(\mathbb{D})$, $f: \mathbb{D} \to \mathbb{D}$

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}.$$

**Def.** Hyperbolic length of a path:

Let $\gamma$ be a piecewise differentiable arc, parametrized by $z(t)$, $t \in [a,b]$. in $\mathbb{D}$

$$\ell_H(\gamma) = \int_a^b \frac{2|z'(t)|}{1-|z(t)|^2}\,dt = \int_\gamma \frac{2|dz|}{1-|z|^2}$$

**Restatement of Schwarz-Pick:**

For any curve $\gamma \subset \mathbb{D}$ and any $f: \mathbb{D} \to \mathbb{D}$ - analytic $\ell_H(f \circ \gamma) \le \ell_H(\gamma)$. If the equality is reached for one curve, then $f$ is Möbius. If $f$ is Möbius, then $\forall \gamma$: $\ell_H(f \circ \gamma) = \ell_H(\gamma)$.

**Proof.** $$\frac{|f'(z(t))|\,|z'(t)|}{1-|f(z(t))|^2} \le \frac{|z'(t)|}{1-|z(t)|^2}.$$ Equality for one point $\Leftrightarrow$ equality for all points $\Leftrightarrow$ $f$ is Möbius. ∎

**Def.** Hyperbolic distance between $z_1, z_2$:

$$d_H(z_1,z_2) = \inf_{\substack{\gamma \text{ from } z_1 \\ \text{to } z_2}} \ell_H(\gamma)$$

**Theorem** $$d_H(z_1,z_2) = \log \frac{1+ \overbrace{\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|}^{\rho(z_1,z_2)}}{1-\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|} = \operatorname{arctanh} \rho(z_1,z_2)$$

The shortest $\gamma$: the arc of circle orthogonal to $\{|z|=1\}$, joining $z_1$ and $z_2$.


**Proof.** Everything (LHS, RHS, circles orthogonal to $\{|z|=1\}$) are Möbius invariant.

So we can map $z_1$ to $0$, $z_2$ to a positive number $r>0$.

Consider any path $\gamma$ from $0$ to $r$, $z(t)=x(t)+iy(t)$, $z:[0,1]\to\mathbb{D}$.

$$\int_0^1 \frac{|z'(t)|}{1-|z(t)|^2}\,dt \geq \int_0^1 \frac{|x'(t)|}{1-x(t)^2}\,dt \geq \int_0^1 \frac{x'(t)}{1-x(t)^2}\,dt = \log \frac{1+x(t)}{1-x(t)}\Big|_{t=0}^{1} = \log\frac{1+r}{1-r}$$

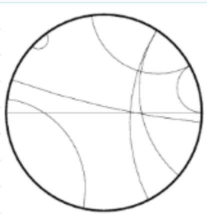
with equality reached exactly when $y\equiv 0$ ($y'\equiv 0$) and $|x'(t)| = x'(t)$, i.e. when $\gamma = [0,r]$, travelled once $\blacksquare$

Hyperbolic geometry:

Points = points in $\mathbb{D}$

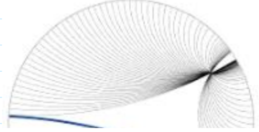
Lines = circular arcs or intervals orthogonal to $\{|z|=1\}$.

Poincare disk model of hyperbolic geometry:

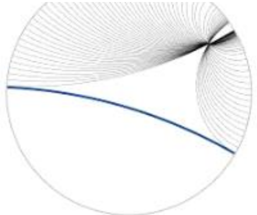


Henri Poincaré

Satisfies all Euclidean Axioms except for parallels:



1. Any two points can be joined by a straight line. (This line is unique given that the points are distinct)
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.



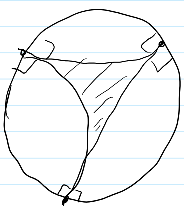
points are distinct)
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. Through a point not on a given straight line, one and only one line can be drawn that never meets the given line.

Spherical geometry. Can be defined the same way on $\hat{\mathbb{C}}$:

$\ell_S(\gamma) = \int_\gamma \frac{2|dz|}{1+|z|^2}$  $d_S(z_1, z_2) = \inf_{\gamma \text{ from } z_1 \text{ to } z_2} \ell_S(\gamma)$ — the same spherical metric!

Also satisfies all Euclidean Axioms except for parallels: lines are great circles, so there are no parallels!

| Geometry | Euclidean | Spherical | Hyperbolic |
|---|---|---|---|
| Infinitesimal length | $\lvert dz\rvert$ | $\frac{2\lvert dz\rvert}{1+\lvert z\rvert^2}$ | $\frac{2\lvert dz\rvert}{1-\lvert z\rvert^2}$ |
| Oriented isometries | $e^{i\varphi}z+b$ | rotations | conformal self-maps |
| Curvature | 0 | $+1$ | $-1$ |
| Geodesics | lines | great circles | circles $\perp$ unit circle |
| Angles of triangle | $=\pi$ | $>\pi$ | $<\pi$ |



$\underbrace{z^2 - x^2 - y^2 = 0}$